

RECOGNIZING PRODUCTS OF SURFACES AND SIMPLY CONNECTED 4-MANIFOLDS

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ABSTRACT. We give necessary and sufficient conditions for a closed smooth 6-manifold N to be diffeomorphic to a product of a surface F and a simply connected 4-manifold M in terms of basic invariants like the fundamental group and cohomological data. Any isometry of the intersection form of M is realized by a self-diffeomorphism of $M \times F$.

1. INTRODUCTION

Simply-connected closed 6-manifolds were classified by Wall [13], Jupp [6], and Žubr [14]. However, if the fundamental group is non-trivial, such complete information is not within reach of current techniques except in special cases.

In this paper we consider the following problem: given a closed, oriented 6-manifold N , can we identify a closed, oriented surface F and a simply-connected closed 4-manifold M such that N is diffeomorphic to $M \times F$? Since simply connected manifolds are already classified, we assume from now on that $F \neq S^2$ has genus ≥ 1 , but the results remain true in the simply connected case. First we discuss some of the necessary conditions.

Condition 1. *The fundamental group $\pi_1(N)$ has to be the fundamental group of a closed, oriented surface F .*

We choose a base-point preserving classifying map $u: N \rightarrow F$ for the universal covering. Up to homotopy and choice of base points this is equivalent to choosing an isomorphism $\alpha: \pi_1(N) \rightarrow \pi_1(F)$, where $u_* = \alpha$.

The next condition concerns the second homology group of the universal covering $H_2(\tilde{N})$, which for the product of F with a simply connected 4-manifold M is a trivial module over $\pi_1(N)$ and so we require this:

Condition 2. *$H_2(\tilde{N})$ is a trivial $\pi_1(N)$ -module.*

Under this assumption, it follows from the spectral sequence of the universal covering that $H_2(\tilde{N})$ is a finitely-generated free abelian group.

Finally, we need some information about the *oriented* integral cohomology ring of N and the Pontrjagin class $p_1(N) \in H^4(N)$, under the assumption that $N \approx M \times F$. Let $q_1: M \times F \rightarrow M$ and $q_2: M \times F \rightarrow F$ denote the first and second factor projection maps. Note that the integral cohomology of $M \times F$ is \mathbb{Z} -torsion free, so any map $H^*(M \times F) \rightarrow H^*(N)$ of integral cohomology rings reduced mod 2 induces a map on $\mathbb{Z}/2$ -cohomology.

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Condition 3. *There exists an isomorphism*

$$\phi: H^*(M \times F) \rightarrow H^*(N)$$

of oriented integral cohomology rings, for some closed, oriented, simply-connected topological 4-manifold M with $KS(M) = 0$. We assume that

- (i) $\phi([M] \times [F]) = [N] \in H^6(N)$,
- (ii) $\phi \circ q_2^* = u^*: H^*(F) \rightarrow H^*(N)$, and
- (iii) ϕ preserves the second Stiefel-Whitney class:

$$\phi(w_2(M \times F)) = w_2(N) \in H^2(N; \mathbb{Z}/2).$$

- (iv) *Moreover, the relation*

$$\langle \phi(x) \cup p_1(N), [N] \rangle = \begin{cases} 0 & \text{if } x = q_1^*(y) \in H^2(M \times F), \\ 3 \operatorname{sign}(M) & \text{if } x = q_2^*([F]) \end{cases}$$

holds for all $y \in H^2(M)$.

In this statement, $KS(M)$ denotes the Kirby-Siebenmann invariant of M , and its vanishing is a necessary and sufficient condition for $M \times F$ to be smoothable (see [7]). If M is a spin manifold, then this condition is assured by requiring $\operatorname{sign}(M) \equiv 0 \pmod{16}$. From now on, we fix a smooth structure and the product orientation on $M \times F$, with respect to given orientations on M and F .

Now we are ready to formulate our main result.

Theorem A. *Let N be a closed, oriented smooth 6-manifold, and $\alpha: \pi_1(N) \cong \pi_1(F)$ for some closed, oriented surface F . Suppose that*

- (i) $H^2(\tilde{N})$ is a trivial $\pi_1(N)$ -module,
- (ii) M is a closed, simply-connected topological 4-manifold with $KS(M) = 0$, and
- (iii) Condition 3 holds for $\phi: H^*(M \times F) \xrightarrow{\cong} H^*(N)$ an oriented ring isomorphism.

Then, there is an orientation and base-point preserving diffeomorphism $f: N \rightarrow M \times F$ such that $f_\# = \alpha$ and $f^ = \phi$.*

Note that the set of distinct smoothings of $M \times F$ is in bijection with $H^3(M \times F; \mathbb{Z}/2)$ by [7]. Theorem A shows that $\operatorname{Homeo}(M \times F)$ acts transitively on the set of smoothings.

Corollary B. *Let M be a closed, simply-connected topological 4-manifold with $KS(M) = 0$, and let F be a closed, oriented surface. Then $M \times F$ has a unique smooth structure.*

We can also ask which automorphisms of the second (co)homology of $M \times F$ are induced by self-diffeomorphisms. In particular, we consider automorphisms of $H^2(M)$, and extend them by the identity on $H^2(F)$ via the identification:

$$(q_1^*, q_2^*): H^2(M) \oplus H^2(F) \cong H^2(M \times F).$$

From the ring structure in cohomology, a necessary condition is that the automorphism on $H^2(M)$ is an isometry of the intersection form.

Corollary C. *Let M be a closed topological 4-manifold with $KS(M) = 0$ and F a closed, oriented surface. Then each isometry of the intersection form of M is induced by a self-diffeomorphism of $M \times F$.*

Remark 1.1. This is interesting, since even in the case where M is itself smooth Donaldson theory (see [5, Theorem 6]) provides examples of isometries of $H^2(M)$ which cannot be realized by self-diffeomorphisms of M .

The conditions in Theorem A can be made more explicit (for checking purposes), and given in a form which explains how the choice of 4-manifold is determined by N . We first remark that Condition 2, and the Serre spectral sequence for the fibration over $F = K(\pi_1(N), 1)$ with fibre \tilde{N} , implies that we have an exact sequence

$$0 \rightarrow H^2(F) \xrightarrow{u^*} H^2(N) \xrightarrow{p^*} H^2(\tilde{N}) \rightarrow 0,$$

where p is the universal covering projection.

Definition 1.2. We abbreviate $V := H^2(N)/u^*H^2(F) \cong H^2(\tilde{N})$, and let $H = \pi_2(N) = H_2(\tilde{N})$.

We have $H^2(\tilde{N}) \cong \text{Hom}_{\mathbb{Z}}(H_2(\tilde{N}), \mathbb{Z})$, so that $H \cong \text{Hom}_{\mathbb{Z}}(V, \mathbb{Z}) = V^*$. The following remark is immediate from the definitions.

Lemma 1.3. *If N satisfies Condition 3 with respect to $M \times F$, then $H^2(M) \cong V$.*

The symmetric bilinear form

$$I(N) : H^2(N)/u^*H^2(F) \times H^2(N)/u^*H^2(F) \rightarrow \mathbb{Z}$$

mapping x and y to $\langle u^*([F]) \cup x \cup y, [N] \rangle$ is then unimodular, where $[F] \in H^2(F)$ is the cohomology fundamental class. Note that this form vanishes on the image of u^* and so is well defined. Thus we have the necessary condition:

Condition 4. *The bilinear form $I(N)$ on $V := H^2(N)/u^*H^2(F)$ is unimodular, and $\text{sign}(I(N)) \equiv 0 \pmod{16}$ if $I(N)$ is even.*

If this condition is fulfilled, there is a *unique* closed, simply-connected topological 4-manifold M with this intersection form and $KS(M) = 0$ by Freedman [4, Theorem 1.5]). This is the starting point for checking the other conditions in Theorem A.

2. THE NORMAL 2-TYPE AND NORMAL 2-SMOOTHINGS

For the proof we use the methods from [8] and assume that the reader is familiar with the basic concepts and theorems although we repeat the relevant definitions briefly. Recall our notation $V = H^2(N)/u^*H^2(F) \cong H^2(\tilde{N})$, and its \mathbb{Z} -dual: $H = \text{Hom}_{\mathbb{Z}}(V, \mathbb{Z}) = \pi_2(N)$. The first step is the determination of the normal 2-type of N . This is a fibration B over BSO where the homotopy groups of the fibre vanish in degree ≥ 3 and such there is a lift of the normal Gauss map of N over B , which is a 3-equivalence. We have to distinguish two cases, where the symmetric bilinear form $I(N) : V \times V \rightarrow \mathbb{Z}$ is even or odd. In the first case, the normal 2-type is

$$p_{\text{even}}B = K(H, 2) \times F \times BSpin \rightarrow BSO,$$

where the map is the composition of the projection to $BSpin$ and the canonical projection to BSO . If the form $I(N)$ is odd, one chooses a characteristic element $v \in V$, and a complex line bundle L_v over $K(H, 2)$ with first Chern class v . Then the normal 2-type is

$$p_{\text{odd}}B = K(H, 2) \times F \times BSpin \rightarrow BSO,$$

where p_{odd} is the map given by the projection to $K(H, 2) \times BSpin$ composed by the map given by the Whitney sum of line bundle L_v and the canonical map to BSO (of course, we have to replace this map by a fibration).

Lemma 2.1. *The normal 2-types of $M \times F$ and N are given by $p_{\text{even}}B$, if M is spin, or $p_{\text{odd}}B$ if M is non-spin.*

Proof. We first look at the second stage of the Postnikov tower of N , this is a fibration over $K(\pi_1(N), 1)$ with fibre $K(\pi_2(N), 2)$, where in our situation $\pi_2(N) = H$. These fibrations are classified by the action of $\pi_1(N)$ on $\pi_2(N)$ and the k -invariant $k \in H^3(\pi_1(N); \pi_2(N))$. This group is zero, and so the action of $\pi_1(N)$ on $\pi_2(N)$ determines the Postnikov tower. If the $\pi_1(N)$ -action is trivial, then we have the trivial fibration. Next, we use our data to construct a 3-equivalence

$$c_{M \times F} := g_{M \times F} \times h_{M \times F}: M \times F \rightarrow K(H, 2) \times F,$$

and a 3-equivalence

$$c_N := g_N \times h_N: N \rightarrow K(H, 2) \times F,$$

which is compatible with our data α and ϕ . For this we consider the map

$$g_{M \times F}: M \times F \rightarrow K(H, 2)$$

such that $(g_{M \times F})^*: V \rightarrow H^2(M \times F) = H^2(M) \oplus H^2(F) = V \oplus H^2(F)$ is the inclusion onto the first summand (see Lemma 1.3), and choose a base point preserving map $g_N: N \rightarrow K(H, 2)$ such that $(g_N)^* = \phi \circ (g_{M \times F})^*$. Then we consider the projection $h_{M \times F} = q_2: M \times F \rightarrow F$ and $h_N = u: N \rightarrow F$. From Conditions 1 - 3 it is clear that the maps $c_{M \times F}$ and c_N are 3-equivalences, with $(c_N)^* = \phi \circ (c_{M \times F})^*$.

If N is Spin-manifold, then by assumption $M \times F$ is a Spin-manifold and we equip both manifolds with an arbitrary Spin structure ω_N and $\omega_{M \times F}$. If N and so $M \times F$ are not Spin-manifolds, then we choose a primitive class $v \in H^2(M \times F; \mathbb{Z})$, such that its component in $H^2(F; \mathbb{Z})$ is zero, which reduces to $w_2(M \times F)$ and a spin structure $\omega_{M \times F}$ on $\nu(M \times F) \oplus L_v$, where L_v is the complex line bundle classified by v . Similarly, we choose a Spin structure ω_N on $\nu(M) \oplus L_{\phi(v)}$. The maps $c_{M \times F}$ and c_N together with the (twisted) Spin-structures are normal 2-smoothings in B . \square

3. THE BORDISM GROUPS

The next step in the proof of Theorem A is to show that, under the given conditions, the normal 2-smoothings constructed in Section 2 are bordant in $\Omega_6(B)$.

To compute the bordism groups we consider the functor associating to a space X the bordism group of $p_{\text{odd/even}}: X \times K(H, 2) \times BSpin \rightarrow BSO$, where the maps are defined as above in the case $X = F$. This is a homology theory denoted by $h_k(X)$ and so we can

use the Mayer-Vietoris sequence to compute it, by writing a surface of genus g as $D_2 \cup Y$, where Y is a wedge of $2g$ circles. Then we obtain an exact sequence

$$\tilde{h}_7(S^2) \rightarrow \tilde{h}_6(Y) \rightarrow \tilde{h}_6(F) \rightarrow \tilde{h}_6(S^2) \rightarrow \tilde{h}_5(Y),$$

or, if we apply the suspension isomorphism, the exact sequence:

$$(3.1) \quad h_5(pt) \rightarrow \sum_{2g} h_5(pt) \rightarrow \tilde{h}_6(F) \rightarrow h_4(pt) \rightarrow .$$

The map from $h_6(F)$ to $h_4(pt)$ is defined by sending $[N, c_N] \mapsto [Q, c_Q]$, where $c_N: N \rightarrow B$ is a lift of the normal Gauss map, and $Q \subset N$ is the pre-image of a regular value of the composition of the map to B with the projection to F . The reference map $c_Q: Q \rightarrow B$ is given by the restriction of c_N to $K := K(H, 2)$, together with the induced bundle and (twisted) Spin-structure.

To proceed further we need information about $h_k(pt)$.

Lemma 3.2. *$h_5(pt)$ is zero and the map $h_6(pt) \rightarrow H_6(K) \oplus H_2(K)$ given by the image of the fundamental class and the image of the Poincaré dual of the first Pontrjagin class is injective. And the map given by the signature and the image of the fundamental class is an injection $h_4(pt) \rightarrow \mathbb{Z} \oplus H_4(K)$.*

Proof. We begin with the computation of $\Omega_k^{\text{Spin}}(\mathbb{CP}^\infty)$ and $\Omega_k^{\text{Spin}}(\mathbb{CP}^\infty; L)$ for $k = 4$ and 6 , where L is the Hopf bundle. The E^2 -term of the Atiyah-Hirzebruch spectral sequence computing $\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty)$ gives \mathbb{Z} in position $(0, 4)$ and $(4, 0)$, and $\mathbb{Z}/2$ in position $(2, 2)$. The differential $d: H_4(\mathbb{CP}^\infty; \mathbb{Z}) \rightarrow H_2(\mathbb{CP}^\infty; \mathbb{Z}/2)$ is the reduction mod 2 composed by the dual of Sq^2 [12, Proposition 1, p. 750] and so is nontrivial. This implies that

$$\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty) \rightarrow \mathbb{Z} \oplus \mathbb{Z}$$

given by the signature and the image of the fundamental class is injective.

Analyzing the Atiyah-Hirzebruch spectral sequence for $\Omega_6^{\text{Spin}}(\mathbb{CP}^\infty)$ gives an entry \mathbb{Z} at position $(2, 4)$ and $(6, 0)$ and $\mathbb{Z}/2$ at position $(4, 2)$. This time the differential vanishes and so the bordism group is either $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$. It was proven in [9, p. 258] that

$$\Omega_6^{\text{Spin}}(\mathbb{CP}^\infty) = \mathbb{Z} \oplus \mathbb{Z},$$

detected by the image of the fundamental class and the image of the Poincaré dual of the first Pontrjagin class.

Now we consider the twisted (by the line bundle L) bordism groups. We reduce the 4-th bordism group to the untwisted case by using the isomorphism given by taking the transversal preimage of \mathbb{CP}^{N-1} , where we replace \mathbb{CP}^∞ by \mathbb{CP}^N for a large N :

$$\Omega_6^{\text{Spin}}(\mathbb{CP}^\infty) \cong \Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L)$$

(here we use that $\Omega_6^{\text{Spin}} = \Omega_5^{\text{Spin}} = 0$) implying that $\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L) \cong \mathbb{Z} \oplus \mathbb{Z}$ again detected by the image of the fundamental class and the signature.

Finally the computation of $\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L) \cong \mathbb{Z} \oplus \mathbb{Z}$ again detected by the image of the fundamental class and the image of the Poincaré dual of the first Pontrjagin class, follows from the Atiyah-Hirzebruch spectral sequence. This time the E^∞ -term is torsion free in

the 6-line, since the differential $d: H_6(\mathbb{CP}^\infty; \mathbb{Z}) \rightarrow H_4(\mathbb{CP}^\infty; \mathbb{Z}/2)$ is the reduction mod 2 composed by the dual of Sq^2 plus $c_1(L) \cup \dots$ (see again [12, Proposition 1, p. 750]) and so is trivial.

Now we consider the general case. We have to show that the bordism groups are again torsion free. Then the statement follows from the Atiyah-Hirzebruch spectral sequence. We first note that by applying an appropriate isomorphism of H we can assume that $H^* = \mathbb{Z}^r$ and in the twisted case $c_1(L) = (0, \dots, 0, 1)$. With this we write $(\mathbb{CP}^\infty)^r = X \times \mathbf{CP}^\infty$ and compute $\Omega_k^{\text{Spin}}(X \times \mathbb{CP}^\infty)$ and $\Omega_k^{\text{Spin}}(X \times \mathbb{CP}^\infty; L)$ for $k = 4$ and 6 , where $X = (\mathbb{CP}^\infty)^{r-1}$ and L is the Hopf bundle over the last factor. We assume inductively that $\Omega_k(X)$ is torsion free for $k = 4$ and $k = 6$. Using again the transversal preimage of \mathbb{CP}^{N-1} , where we replace \mathbb{CP}^∞ by \mathbb{CP}^N for a large N , we have an exact Gysin sequence (see [2, Section I.6, p. 315], [11]):

$$\Omega_5^{\text{Spin}}(X \times \mathbb{CP}^\infty; L) \rightarrow \Omega_6^{\text{Spin}}(X) \rightarrow \Omega_6^{\text{Spin}}(X \times \mathbb{CP}^\infty) \rightarrow \Omega_4^{\text{Spin}}(X \times \mathbb{CP}^\infty; L).$$

Since the odd dimensional groups are by the Atiyah-Hirzebruch spectral sequence torsion, we see that $\Omega_6^{\text{Spin}}(X \times \mathbb{CP}^\infty)$ is torsion free, if $\Omega_4^{\text{Spin}}(X \times \mathbb{CP}^\infty; L)$ is torsion free. For this we consider the corresponding exact Gysin sequence (again, see [2, Section I.6]):

$$\Omega_3^{\text{Spin}}(X \times \mathbb{CP}^\infty; L \oplus L) \rightarrow \Omega_4^{\text{Spin}}(X) \rightarrow \Omega_4^{\text{Spin}}(X \times \mathbb{CP}^\infty; L) \rightarrow \Omega_2^{\text{Spin}}(X \times \mathbb{CP}^\infty; L \oplus L).$$

The Atiyah-Hirzebruch spectral sequence implies that

$$\Omega_2^{\text{Spin}}(X \times \mathbb{CP}^\infty; L \oplus L) \cong H_2(X \times \mathbb{CP}^\infty) \oplus \mathbb{Z}/2.$$

Now we compare this exact sequence with that for X a point:

$$\Omega_3^{\text{Spin}}(\mathbb{CP}^\infty; L \oplus L) \rightarrow \Omega_4^{\text{Spin}} \rightarrow \Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L) \rightarrow \Omega_2^{\text{Spin}}(\mathbb{CP}^\infty; L \oplus L).$$

We have maps from the first to the second exact sequence given by the projection from X to a point. Now suppose that $\Omega_4^{\text{Spin}}(X \times \mathbb{CP}^\infty; L)$ contains a torsion element. Then, since by assumption $\Omega_4^{\text{Spin}}(X)$ is torsion free, this maps to the non-trivial torsion element in $\Omega_2^{\text{Spin}}(X \times \mathbb{CP}^\infty; L \oplus L)$. But then the image in $\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L)$ is again a non-trivial torsion element, since in $\Omega_2^{\text{Spin}}(\mathbb{CP}^\infty; L \oplus L)$ it maps to the non-trivial element. But this is a contradiction to what we have shown above that $\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L)$ is torsion free.

Now we have shown half of our statements, namely that $\Omega_6^{\text{Spin}}(X \times \mathbb{CP}^\infty)$ is torsion free as well as $\Omega_4^{\text{Spin}}(X \times \mathbb{CP}^\infty; L)$. We prove the other cases by a similar argument using this time the exact Gysin sequences:

$$\Omega_5^{\text{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 2}) \rightarrow \Omega_6^{\text{Spin}}(X) \rightarrow \Omega_6^{\text{Spin}}(X \times \mathbb{CP}^\infty; L) \rightarrow \Omega_4^{\text{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 2})$$

and

$$\Omega_3^{\text{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 3}) \rightarrow \Omega_4^{\text{Spin}}(X) \rightarrow \Omega_4^{\text{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 2}) \rightarrow \Omega_2^{\text{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 3}).$$

This case is easier since $\Omega_2^{\text{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 3})$ is torsion free, the torsion in the E^2 term is killed by the d_2 -differential.

Finally we show that $\Omega_4^{\text{Spin}}(X \times \mathbb{CP}^\infty)$ is torsion free using the exact sequence:

$$\Omega_3^{\text{Spin}}(X \times \mathbb{CP}^\infty; L) \rightarrow \Omega_4^{\text{Spin}}(X) \rightarrow \Omega_4^{\text{Spin}}(X \times \mathbb{CP}^\infty) \rightarrow \Omega_2^{\text{Spin}}(X \times \mathbb{CP}^\infty; L).$$

By the same argument as above the group $\Omega_2^{\text{Spin}}(X \times \mathbb{CP}^\infty; L)$ is torsion free finishing the argument.

Now we show that $h_5(pt) = 0$. On the line corresponding to $h_5(pt)$ the only non-trivial entry in the E_2 -term is $H_4(K; \mathbb{Z}/2)$. If $I(N)$ is even, the differentials are even given by the dual of Sq^2 . If $I(N)$ is odd, where we had to use twisted Spin-structures, the differentials are given by the dual of Sq^2 plus $x \mapsto Sq^2x + w_2 \cup x$, where w_2 is the reduction of c mod 2. It is an easy exercise to show that the E^3 -term is zero in both cases. \square

With this information we show that the bordism classes of N and $M \times F$, equipped with the normal 2-smoothings constructed in Section 2, agree when identified via the maps α and ϕ . By the exact sequence (3.1) and Lemma 3.2, this amounts to showing (i) the bordism classes in $h_6(pt)$ agree, and (ii) that the classes in $h_4(pt)$ agree, which we obtain as transversal preimages of a regular value of the map to F given by composing our normal 2-smoothings with the projection to F .

By Lemma 3.2, the first invariant is given by two invariants, the image of the fundamental class in $H_6(K(H, 2))$ and the image of the Poincaré dual in $H_2(K(H, 2))$. The image of the fundamental class in $H_6(K(H, 2))$ is (by the cohomological structure of $K(H, 2)$) equivalent to the triple product $g^*(x) \cup g^*(y) \cup g^*(z)$ for classes x, y, z in $H^2(K(H, 2))$. But these products vanish for $M \times F$ with $g = g_{M \times F}$, and for N with $g = g_N$, since ϕ is an isometry of the cohomology rings and $(g_N)^* = \phi \circ (g_{M \times F})^*$. The image of the Poincaré dual in $H_2(K(H, 2))$ is determined by the products $g^*(x) \cup p_1$ for all $x \in H^2(K(H, 2))$ and vanishes for $M \times F$ and for N by Condition 3.

Thus we are left with the invariant in $h_4(pt)$. Let $Q \subset N$ be the transversal preimage of a regular value of the map $u: N \rightarrow F$. By Lemma 3.2, bordism classes in $h_4(pt)$ are determined by the signature of the underlying 4-manifold, and the image of the fundamental class in $H_4(K(H, 2))$. For a class $\beta \in H^4(N)$ we have the adjunction formula

$$\langle u^*([F]) \cup \beta, [N] \rangle = \langle i^*(\beta), [Q] \rangle,$$

where $i: Q \rightarrow N$ is the inclusion. Applying this to $\beta = p_1(N)$ we obtain:

$$\langle p_1(N) \cup u^*([F]), [N] \rangle = \langle p_1(Q), [Q] \rangle,$$

since the normal bundle of Q is trivial. The signature theorem for Q and Condition 3 (iv) imply that

$$\langle p_1(N) \cup u^*([F]), [N] \rangle = 3 \text{sign}(Q) = 3 \text{sign}(M),$$

proving the equality for the first invariant in $h_4(pt)$.

For the second invariant we note that the image of the fundamental class of Q in $H_4(K(H, 2))$ is determined by the numbers

$$\langle i^*g^*(x) \cup i^*g^*(y), [Q] \rangle.$$

We apply again the adjunction formula for $\beta = g^*(x) \cup g^*(y)$, and get

$$\langle g^*(x) \cup g^*(y) \cup u^*([F]), [N] \rangle = \langle i^*g^*(x) \cup i^*g^*(y), [Q] \rangle,$$

where $g = g_N$. A similar formula holds for $M \times F$ and $g = g_{M \times F}$. The left side agrees for N and $M \times F$, since ϕ is an isometry of the cohomology ring. Thus also the second invariant for the element in $h_4(pt)$ agrees. Summarizing, we have shown:

Proposition 3.3. *If the conditions of Theorem A are fulfilled, then the bordism classes*

$$[N, c_N] = [M \times F, c_{M \times F}] \in \Omega_6(B),$$

for the normal 2-smoothings on N and $M \times F$ constructed in Lemma 2.1.

4. THE PROOF OF THEOREM A

We consider N and $M \times F$ equipped with normal 2-smoothings compatible with α and β . By Proposition 3.3, the corresponding bordism classes are equal. Choose a B -bordism W between these two normal 2-smoothings. Since the Euler characteristic of N and $M \times F$ agrees there is an obstruction $\theta(W) \in l_7(\pi_1(N))$ which is elementary if and only if W is B -bordant to an s-cobordism. We first note that the Whitehead group for $\pi_1(F)$ vanishes by a result of Farrell-Hsiang [3] so that we can ignore decorations in the l -monoids and L -groups. Next we note that for $F \times N$ the intersection form with values in the group ring on π_3 vanishes identically. By [8, Proposition 8], this implies that $\theta(W)$ sits in the ordinary L -group $L_7(\pi_1(N))$. But by Cappell [1, Theorem 18], there is a closed 7-manifold with B -structure so that after taking the disjoint union of W with this manifold the obstruction in $L_7(\pi_1(N))$ vanishes. This completes the proof.

The proof of Corollary B. We can apply Theorem A to the topological manifold $M \times F$ equipped with two different smoothings. By Novikov [10, Theorem 1], we have Condition 3 with $\phi = id$. \square

The proof of Corollary C. There is an automorphism ϕ of $H^*(M \times F)$, which on $H^2(M \times F)$ is the given isometry on $H^2(M)$ extended by the identity on $H^2(F)$. By Theorem A there is a diffeomorphism inducing ϕ , and in particular the given isometry on $H^2(M)$. \square

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